ABSTRACTS OF INVITED SPEAKERS LECTURES

20th International Workshop for Young Mathematicians "Number Theory"

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This booklet contains abstracts of invited speakers lectures that will be given at Workshop. Each speaker will perform three one-hour lectures.

Jörg Brüdern
Georg-August-Universität Göttingen

Limit periodic functions in number theory

We explore a perhaps unexpected link between the theory of limit period and almost perioc functions on the intergers, and the famous Hardy-Littlewood circle method. The latter, most commonly known as a tool to investigate diophantine equation, is a subtle harmonic analysis on the unit circle in the complex plane (hence the name). Along the way, we will see binary additive problems like the twin problem for square-free numbers, applications to sums of cubes, and a general principle for the distribution of limit-periodic functions in arithmetic progressions.

Prerequisites. Students should have basic knowledge of elementary number theory, and should be aquainted with the language of functional analysis (norms, Banach spaces).

Bartłomiej Bzdęga
Adam Mickiewicz University in Poznań

Lecture 1. The famous *abc* conjecture by Oesterle and Masser asserts that for all coprime integers a, b and c satisfying $a+b+c=0$ we have

$$|a|, |b|, |c| < K_\varepsilon \mathrm{rad}(abc)^\varepsilon,$$

where K_ε is a constant depending only on ε and $\mathrm{rad}(n)$ is the product of primes dividing n.

In early 80s Mason and Stothers proved a variation of this conjecture in which a, b and c are polynomials over $\mathbb{C}$ and the Euclidean norm $|\cdot|$ on $\mathbb{Z}$ is replaced by the Euclidean norm $\deg(\cdot)$ on $\mathbb{C}[x]$. We present a short proof of this theorem by Lang. Additionally we show some applications.

Bibliography: V.V. Prasolov, *Polynomials.*

Lecture 2. For relatively prime positive integers h and k we define the Dedekind sum

$$s(h,k) = \sum_{j=1}^{k-1} \frac{j}{k}\left(\frac{hj}{k} - \left\lfloor \frac{hj}{k} \right\rfloor - \frac{1}{2}\right).$$

This sum arises from the Dedekind functional equation of the η function.

Dedekind sums satisfy the reciprocity law

$$s(h,k) + s(k,h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right).$$

We prove this and present some applications, especially the proof of reciprocity law for the Jacobi symbol or computing the number of lattice points on some tetrahedra.

Bibliography: H.Rademacher and E.Grosswald, *Dedekind Sums.*

Lecture 3. Consider a polygon with all vertices at lattice points. If we wish to compute the number of lattice points on this polygon, we can use the Pick's formula $A = i + \frac{1}{2}b$, where A is the area of the polygon and b and i are the numbers of lattice points respectively on the boundary and on the interior of the polygon.

The problem is much more difficult if we assume that the vertices have rational coordinates. We show how to reduce this problem to the problem of the triangle with vertices $(0,0)$, $(X,0)$ and $(0,Y)$ and solve it for $X = \frac{t}{a}$ and $Y = \frac{t}{b}$ with relatively prime a and b.

The method is to consider the generating function of the sequence $L(t)$ of numbers of lattice points of such triangles. Then we use the Residue Theorem to obtain the proper coefficient.

Bibliography: M.Beck and S.Robins, *Computing the Continuous Discretely.*

Sander Dahmen
Vrije Universiteit Amsterdam

Ternary Diophantine equations

In these lectures we study Diophantine equations with three terms. Special emphasis will be on generalized Fermat equations, i.e. equations of the form $x^p + y^q = z^r$, where $p, q, r > 1$ are given integers, to be solved in coprime integers x, y, z. (Occasionally, some of the exponents p, q, r will also be considered as variables, in which case we speak about exponential Diophantine equations.) After some general theory and an overview of results and open problems, we focus on techniques to explicitly solve a variety of ternary Diophantine equations, including some generalized Fermat equations. The arithmetic of algebraic curves and modular forms will play a key role in this. (Some other important subjects in the area, such as linear forms in logarithms, will only be mentioned very briefly in these lectures.) Since it will take us (way) to far afield to introduce all the mathematical machinery involved from scratch, we will present a few "clean black boxes" in some cases to work with. Next to a theoretical approach, we will also rely on computer algebra packages, e.g. SageMath or Magma, to perform explicit computations.

Florian Luca
Wits, Max-Planck, Ostrava

Diophantine equations

In this short course, we will take a field trip into the world of Diophantine equations, which are equations of several variables to be solved in integers or rational numbers. Specifically, we will look at:

- Quadratic equations (Pythagorean triples, Pell equations).
- Effective methods: Baker's bounds on linear forms in logarithms and applications to Exponential Diophantine Equations.
- Ineffective Methods: The Subspace Theorem and its applications.

I shall also discuss the *abc*-conjecture and present a couple of its applications.

Bartosz Naskręcki
University of Bristol

L-functions and Galois representations

L-functions and Galois representations L-functions and Galois representations are two types of tools which became indispensable in number theory during last few decades. In this mini-course we will explain what is an L-function, describe the most elementary examples and to which objects in number theory they are related. We will also describe the role which Galois representations take in their constructions. Our discussion will be motivated by examples coming from the theory of elliptic curves, modular forms and certain more general constructions related to cohomology groups of algebraic varieties.

First hour: Definition of an L-function. Properties of coefficients, Euler product and functional equation. Basic examples: L-functions of Dedekind, Hecke, Artin and those attached to elliptic curves.

Second hour: Definition of the absolute Galois group. Galois action on torsion points of elliptic curves, Artin representations, representations of cohomology groups of algebraic varieties. Information about the image of certain representations and construction of representations with prescribed image.

Third hour: Information about applications in number theory: Birch-Swinnerton-Dyer conjecture, Last Fermat's Theorem and generalisations, link to modularity of algebraic varieties, special values of L-functions.