Homotopy Theory of Lie groups and their Classifying Spaces

Syllabus of the course, $3 \times 1,5$ hours

Stefan Jackowski
University of Warsaw, Poland

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Module 1 – Lecture 1

1. Lie groups, homomorphisms and linear representations. Irreducible representations.
2. Maximal tori in compact Lie groups.
3. Characters of representations. Ring of virtual characters. The Weyl theorem.
4. Actions of Lie groups. Homogeneous spaces (orbits) and equivariant maps.
5. Classifying spaces of topological groups and maps induced by homomorphisms.
6. Homotopy classification of maps between classifying spaces of discrete groups.
7. The Dwyer–Zabrodsky–Nottbohm theorem on homotopy classification of maps from the classifying spaces of p–toral groups to the classifying space of compact Lie groups.
8. Linear representations vs. homotopy representations of compact Lie groups.

Module 2 – Problem Session

Participants will be split into groups of no more than 15 people, according to their familiarity with the subject. Each group will be working on problems related to Lecture 1 and preparation to Lecture 2. Problems will be announced before the course begins.

Module 3 – Lecture 2

1. Localization and completion in homotopy theory. The Sullivan arithmetic square.
2. Classifying spaces of small topological categories. Homotopy colimits.
3. Decompositions of the classifying spaces of compact Lie groups into homotopy cilimit of the classifying spaces of its subgroups.
4. Obstruction theory.
5. Exotic maps between classifying spaces. Unstable Adam operations.
6. Semi-ring of homotopy representations and its Grothendieck ring.
7. Very exotic maps and open question on homotopy representation theory of compact Lie groups.

Prerequisites

For understanding Lecture 1 and participation in the Problem Session we assume familiarity with basic courses in:

- Topology, including basic notions of homotopy and fundamental groups. (cf. [14])
- Linear Algebra (basic notions, unitary matrices, diagonalization)
- Group Theory (basic notions, Sylow theorem)

Lecture 2 is aimed at more advanced audience. We assume familiarity with Algebraic Topology course (homology, cohomology, higher homotopy groups) and some commutative algebra (localization, completion - cf. [3]).

References

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