Ultrafilters

Applications in ergodic theory

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Filter — family of "large sets", i.e. A — "large" iff $A \in \mathcal{F}$.

Ultrafilter — maximal filter: any set $A \subset \mathbb{N}$ is either "large" $(A \in p)$ or "small" $(A^c \in p)$.

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Fix $a \in \mathbb{N}$. Declare $A \in \mathcal{F}$ iff $a \in A$. Then $\mathcal{F}$ is a filter, and even an ultrafilter.

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Lemma

1. *A filter is an ultrafilter if and only if it is a maximal filter.*
2. *Any filter $\mathcal{F}$ can be extended to an ultrafilter p with $\mathcal{F} \subset p$.*
3. *There exist ultrafilters which are not principal.*

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Warning: Care needed for *non*-commutative semigroups — but can be done!

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Extra structure II.

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For $A \in \mathcal{P}(\mathbb{N})$, declare $\bar{A} \subset \beta(\mathbb{N})$ to be the set:

$$\bar{A} = \{p \in \beta(\mathbb{N}) \ : \ A \in p\}.$$

We endow $\beta(\mathbb{N})$ with the topology generated by $\bar{A}$ (for $A \in \mathcal{P}(\mathbb{N})$) as the basis of open sets. (i.e. open sets = unions of $\bar{A}$'s)

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- **Left-topological semigroup:**
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  - The map $p \mapsto p + q$ is continuous.
- **Extension of** $\mathbb{N}$ — for $n \in \mathbb{N}$ form a principal ultrafilter $\hat{n} = \{A \in \mathcal{P}(\mathbb{N}) \ : \ n \in A\}$. Then, the map:

  $$\mathbb{N} \ni n \mapsto \hat{n} \in \beta(\mathbb{N})$$

  is an isomorphism. We can (and will) pretend that $\mathbb{N} \subset \beta(\mathbb{N})$.

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- **Not** constructive: consistent with ZF that no non-trivial ultrafilters exist.
- *"Three-headed monster"* (Jan van Mill).

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3. If $\mathcal{F}$ = cofinite subsets of infinite $L \subset \mathbb{N}$, then $\mathcal{F}\text{-}\lim_n x_n = \lim_{n\to\infty,\ n\in L} x_n$.

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1. *If p is an ultrafilter on $\mathbb{N}$, Z is a compact Hausdorff space, and $x_n \in Z,\ n \in \mathbb{N}$ then the generalised limit*

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2. *The map $x \mapsto p\text{-}\lim_n x_n$ preserves coordinatewise operations, i.e.:*
   - $p\text{-}\lim_n (x_n + y_n) = (p\text{-}\lim_n x_n) + (p\text{-}\lim_n y_n)$
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3. *The generalised limits and the algebraic structure are related by:*

$$(p+q)\text{-}\lim_n x_n = p\text{-}\lim_m q\text{-}\lim_n x_{n+m}.$$

In particular, if $p + p = p$ (idempotent), then

$$p\text{-}\lim_n x_n = p\text{-}\lim_m p\text{-}\lim_n x_{n+m}.$$

Finite sums set: Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. Then define:

$$\mathrm{FS}(x) = \left\{ \sum_{i\in I} x_i \;:\; I \subset \mathbb{N},\ 0 < \#I < \infty \right\}$$

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1. A — IP-set iff for some sequence x we have $\mathrm{FS}(x) \subset A$.
2. A — IP*-set iff for any IP-set B we have $A \cap B \neq \emptyset$.

Theorem (Galvin–Glazer)

Let $A \subset \mathbb{N}$. The following are equivalent:

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Rotations on a circle.

A toy model.

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Dynamical system under consideration:

1. Space: torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq \{|z| = 1\}$.
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Special case: if $\alpha_k = c_k \alpha$ for $k \geq 1$, $c_k \in \mathbb{Z}$ then

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Let $f : \mathbb{N} \to \mathbb{T}$ *be a polynomial,* $f(0) = 0$*. Let* $p \in \beta(\mathbb{N})$ *be idemptent (i.e.* $p + p = p$*). Then:*

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Real polynomials: Let $g : \mathbb{R} \to \mathbb{R}$ be a polynomial, $g(0) = 0$. Then the set $\{n \in \mathbb{N} \ : \ \mathrm{dist}(g(n), \mathbb{Z}) < \varepsilon\}$ is IP^*.

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$$p\text{-}\lim_{n} f(n) = p\text{-}\lim_{n} p\text{-}\lim_{m} f(n+m) \qquad \text{(idempotence)}$$

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Hence, $p\text{-}\lim_n f(n) = 0$, Q.E.D.

$\square$

Dynamical systems.

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Recall: Measure preserving system $\mathbf{X}$ constists of:

1. Compact topological space X.
2. σ-algebra $\mathcal{M}$ and probability measure μ.
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Return times: Take $A \in \mathcal{M}$, $\mu(A) > 0$. Consider the set:

$$E_\varepsilon = \left\{ n \in \mathbb{N} \ : \ \mu(A \cap T^{-n} A) > \mu(A)^2 - \varepsilon \right\}$$

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General question: *How large is E_ε?* (for given $\mathbf{X}$ and A)
E.g. infinite? syndetic? IP^*? etc.

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$$\begin{aligned} p\text{-}\lim_n \mu(A \cap T^{-n}A) &= p\text{-}\lim_n \langle 1_A, U_T^n 1_A \rangle = \langle 1_A, P 1_A \rangle \\ &= \|P1_A\|^2 \|1_X\|^2 \geq \langle P1_A, 1_X \rangle^2 = \langle 1_A, P1_X \rangle^2 = \mu(A)^2 \end{aligned}$$

Hence, for p-many n's: $\mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon$.

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Theorem (Bergelson, Fustrenberg & McCutcheon; Schnell)

Let $h \in \mathbb{Z}[x]$, $h(0) = 0$, and let $p \in \beta(\mathbb{N})$ be idempotent. Then $p\text{-}\lim_n U_T^{h(n)}$ is a projection.

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Corollary

If F is an IP *set, h is a polynomial with $h(0) = 0$, then $h(F) \cap E_\varepsilon \neq \emptyset$. Equivalently, for a given $h \in \mathbb{Z}[x]$, $h(0) = 0$, the following set is* IP**:*

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Let $A \subset \mathbb{N}$ *be a set with positive Banach density* $d^*(A) > 0$*, and let* $h \in \mathbb{Z}[x]$, $h(0) = 0$*. The set following set is* IP^**:*

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Further research:

1. Multiple recurrence or many transformations. E.g. $\left\{n \in \mathbb{N} \ : \ \mu(A \cap T_1^{-n} A \cap \dots T_k^{-n} A) > c\right\}$ is IP^* (even IP_r^*).

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3. C-sets (central) and D-sets; minimal and essential idempotents.

Thank You
for your attention!