The simplex of invariant measures

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Generally about invariant measures

(X, T) — topological dynamical system
(X compact metric, $T : X \to X$ continuous)

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$$\mathrm{ex}\mathcal{M}_T(X) = \mathcal{M}_T^{\mathrm{e}}(X)$$

- moreover, it is a *Choquet simplex* (see e.g. [3] Walters 1982)

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If $K = \mathcal{M}_T(X)$ and $\mu \in \mathcal{M}_T(X)$ then

$$\mu = \int_{\mathcal{M}_T^{e}(X)} \nu \, d\xi_\mu(\nu)$$

(*the ergodic decomposition*).

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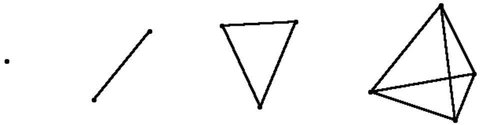

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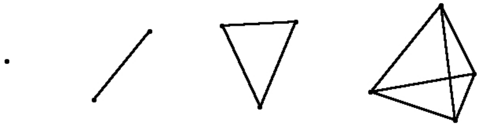

If $\mathrm{ex}K$ is closed (i.e. compact) then $K \approx \mathcal{M}(\mathrm{ex}K)$ (*Bauer simplex*). Finite-dimensional simplices are Bauer.

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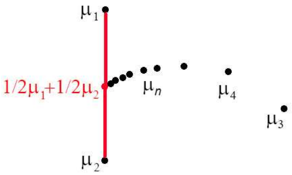

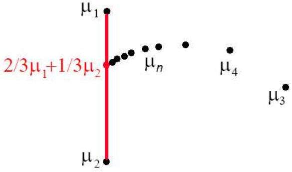

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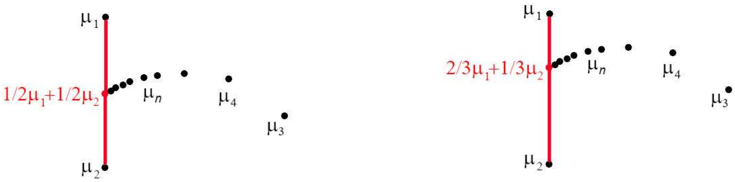

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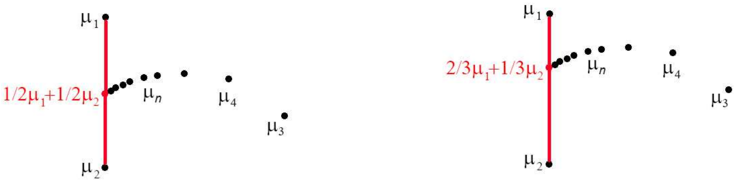

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Proof very hard...

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A few easier facts will be left as exercises.

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Theorem 2 ([11])

For every Choquet simplex K there exists a (non-minimal) subshift (X, T) for which $\mathcal{M}_T(X) \approx K$ and which has the dyadic odometer as a factor.

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- Exercise 1: Let π_n be a uniformly convergent sequence of homeomorphisms of a compact space into a metric space. If, for each n, the uniform distance between π_n and π_{n+1} is small enough (the bound depends on the properties of π_n), then the limit map is also a homeomorphism.

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- A Borel probability measure μ on $\Lambda^{\mathbb{Z}}$ is *invariant* if $\mu(A) = \mu(T^{-1}(A))$ for every Borel set A, where T is the shift map $T(x)_n = x_{n+1}$. In particular

$$\mu(B) = \mu(\square B) = \sum_{a\in\Lambda} \mu(aB).$$

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- Exercise 4: This metric is *convex* (the balls are convex). So, we are situated in a *locally convex space*.

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For every $\epsilon > 0$ there exists N such that whenever $B = B_1 B_2 \dots B_k$, where $|B_1| = |B_2| = \dots = |B_k| \geq N$ (and $k \geq 1$ is arbitrary), then

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Proof: Exercise 5.

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Conversely, every *ergodic* measure on X is approximated by some blocks occurring in X.

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The *dyadic odometer* (G, τ) is the subshift over the countable (compact) alphabet $\mathbb{N}_0 \cup \infty$, consisting of sequences following the rule: every second term is 1, every fourth term is 2, ..., every 2^kth term is k. At most one term ∞ is allowed.

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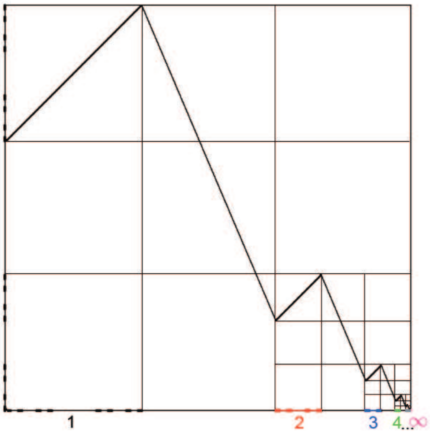
The odometer (G, τ) is conjugate to the interval map shown on the following figure, restricted to the classical Cantor set (which is invariant and on which the map is a homeomorphism).

Odometers



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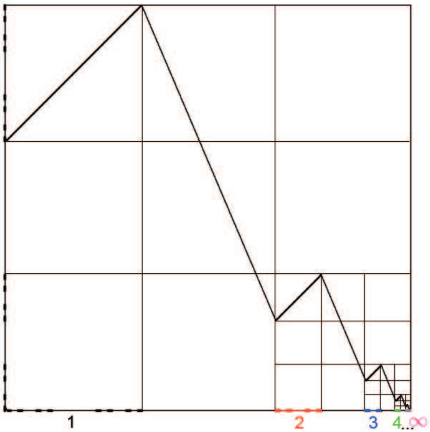

The odometer is a rotation of a compact monothetic group, hence it is *minimal* and *uniquely ergodic*.

The measure on G

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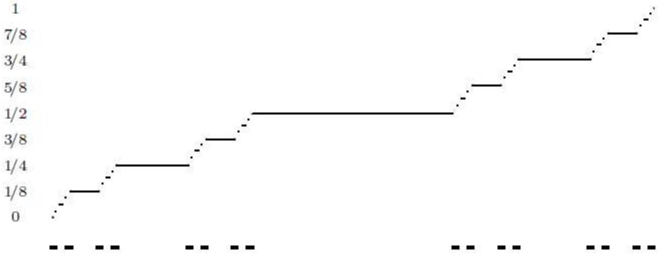

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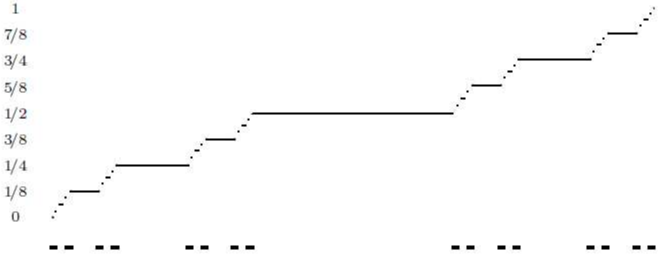

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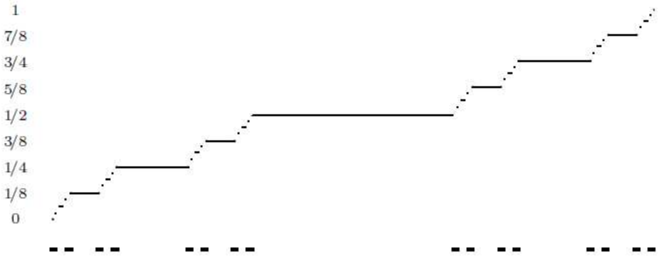

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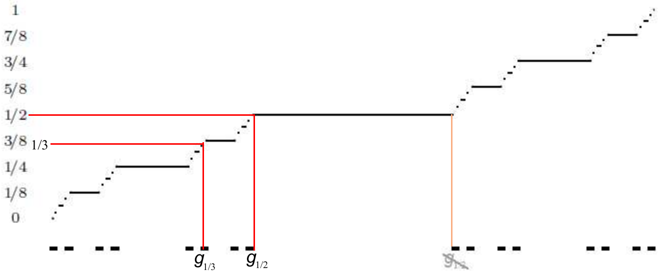

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For $t \in [0, 1]$ we define $g_t = \min(F_\lambda^{-1}(t))$.

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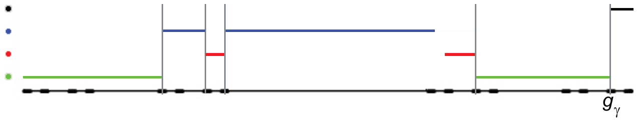

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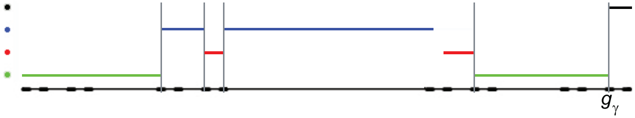

We will identify semicocycles which differ only at the discontinuities. In such space we apply the L^1-distance:

$$d_1(f, h) = \lambda\{g : f(g) \neq h(g)\}.$$

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Moreover, only countably many points $g \in G$ produce multiple points x, hence (X_f, T) is uniquely ergodic and isomorphic to (G, λ, τ). We will denote the unique invariant measure of (X_f, T) by μ_f.

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The map $f \mapsto \mu_f$ is injective and continuous (d_1 vs. d^*).

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It is an easy observation that the set $\{g : f(g) = h(g)\}$ is relatively both open and closed within the set where both functions are continuous. Hence, by adding or subtracting at most countably many points, at which either f or g is discontinuous (hence not belonging to the orbit of 0) we can make this set either open or closed, without changing its measure.

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Plugging this into the formula for $d^*(\mu_f, \mu_h)$ we get $d^*(\mu_f, \mu_h) \leq d_1(f, h)$. $\square$

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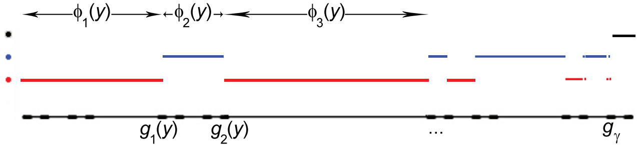

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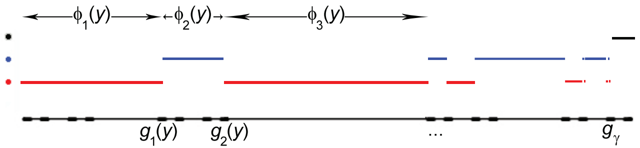

Clearly, the map $y \mapsto f_y$ is continuous in d_1, moreover, different points yield essentially different semicocycles. By compactness of Y, this is a homeomorphism. □

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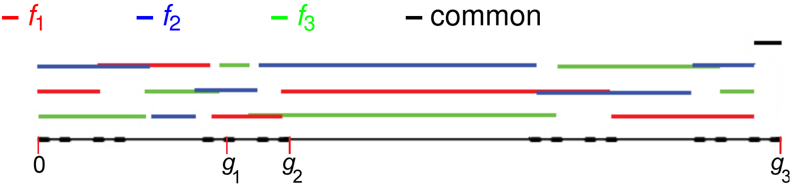

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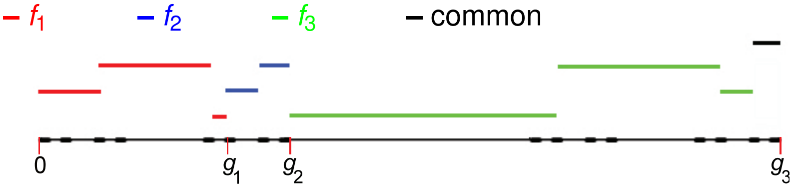

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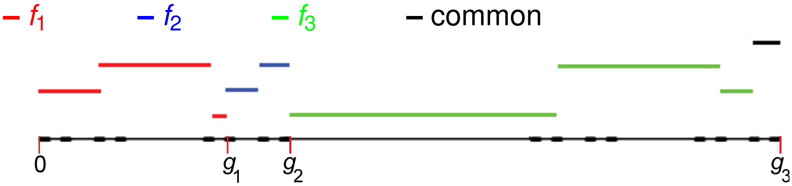

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Clearly, this is again a semicocycle and its unique invariant measure depends continuously on the coefficients p_i.

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For N of the form 2^j we also define the *Nth order mixture of* (f_i) with coefficients (p_i). Namely, we divide the odometer into N equal parts (each homeomorphic to the whole) and we apply the mixture on each part separately (see the figure for $N = 4$).

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By **Lemma 1**, B is close to the arithmetic average of the measures represented by the involved N-blocks, hence to the combination of μ_{f_i} with coefficients p_i. $\square$

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Inductive step: Suppose we have defined a homeomorphic embedding $y \mapsto f_y^{(n)}$ on $\mathrm{ex}B_n$ and extended it to an affine homeomorphism π_n on B_n onto the simplex B_n' (of measures) spanned by $\{\mu_{f_y^{(n)}} : y \in \mathrm{ex}B_n\}$. By change of the metric in B_n we can assume that π_n is an isometry.

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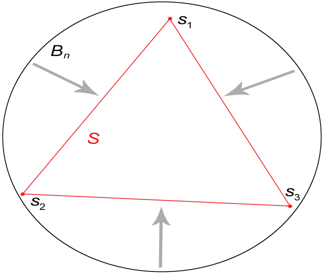

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Hint: On a compact domain, if $f > 0$ is lower semicontinuous and (f_n) is a sequence of continuous functions increasing to f, then $f_n > 0$ for some n.

Let $f_i = f_{s_i}^{(n)}$ ($i = 1, \dots, k$). The composition $\pi_n \circ s$ sends B_n continuously onto the simplex spanned by $\{\mu_{f_i}, i = 1, \dots, k\}$ and this map is ϵ_n-close to π_n. We have

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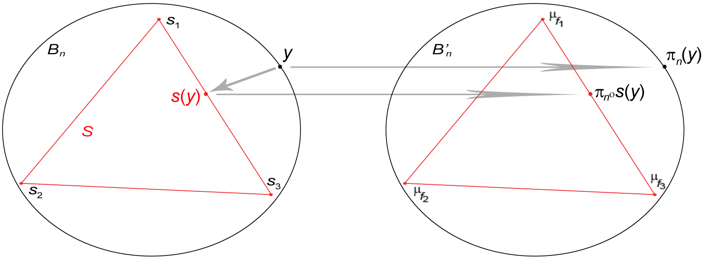

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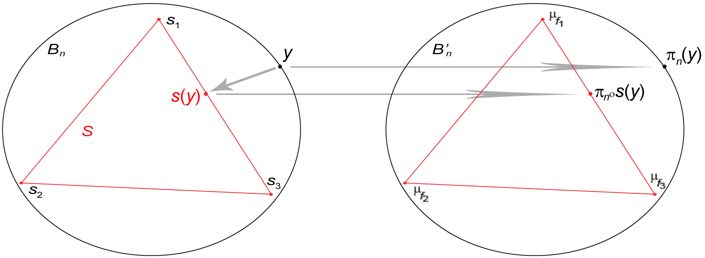

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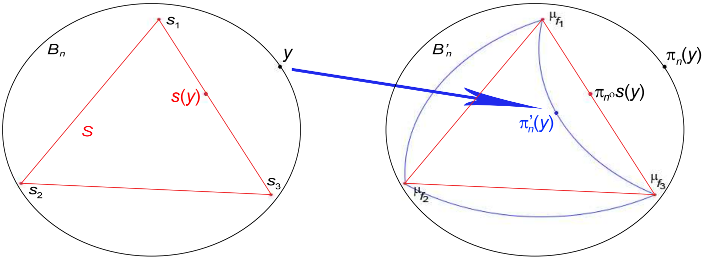

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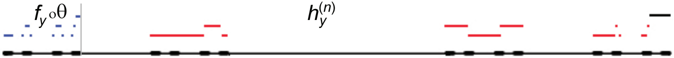
- We define

$$f_y^{(n+1)}(g) = \begin{cases} f_y \circ \theta(g) & g \in G_0, f_y \circ \theta(g) \neq 2 \\ 0 & g \in G_0, f_y \circ \theta(g) = 2 \\ h_y^{(n)}(g) & g \notin G_0. \end{cases}$$



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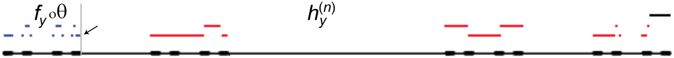

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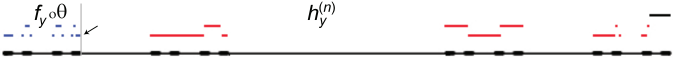

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So defined map $y \mapsto f_y^{(n+1)}$ is *injective* and continuous on $\mathrm{ex}B_{n+1}$ thus it is a homeomorphic embedding.

If N is large enough then $d_1(f_y^{(n+1)}, h_y^{(n)}) < \epsilon_n$. Thus $d^*(\mu_{f_y^{(n+1)}}, \pi'_n(y)) < \epsilon_n$. Letting $\pi_{n+1}(y) = \mu_{f_y^{(n+1)}}$, we get

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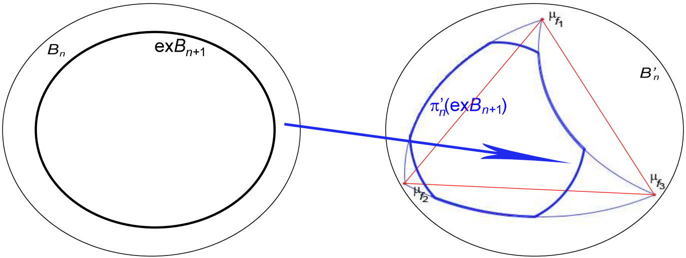

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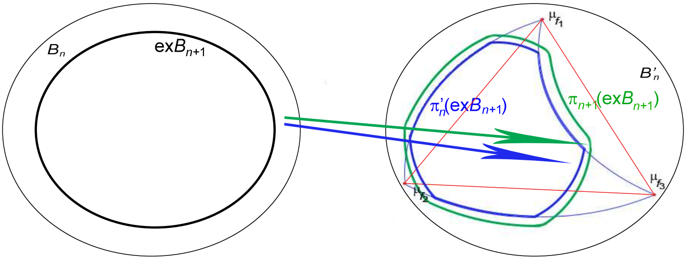

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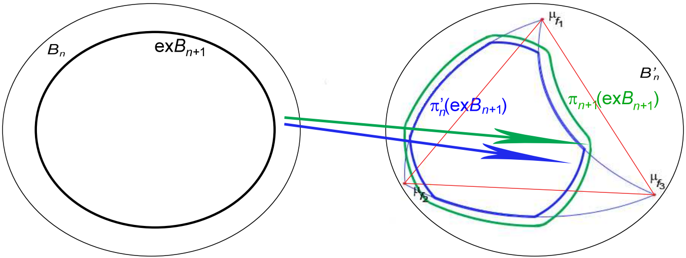

End of induction.

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We have also defined maps π'_n acting on B_n (hence also on K) by assigning semicocycles $h_y^{(n)}$ (and then taking the corresponding measures). These maps are neither injective nor affine, yet **they converge to the same limit map π_∞.**

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We define the target subshift space as

$$X = \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} \bigcup_{y\in K} X_{h_y^{(m)}}} \quad \left(= \bigcap_{n\geq 1} X_n\right).$$

It remains to show that $\mathcal{M}_T(X) = \pi_\infty(K)$.

Fix some $y \in K$ and consider two integers $m > n$. The measure $\pi'_m(y)$ is supported by $X_{h_y^{(m)}}$, hence also by the larger closed set X_n. This implies that $\pi_\infty(y)$ is also supported by X_n. Since this is true for each n, $\pi_\infty(y)$ is supported by X, i.e., we have shown that $\pi_\infty(K) \subset \mathcal{M}_T(X)$.

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For the converse, it suffices to show that any *ergodic* measure μ on X is in $\pi_\infty(K)$. Fix n and let N be the parameter used in the nth construction step. Now the assertions of **Lemma 1** and **Lemma 2** are fulfilled up to ϵ_n.

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Let B be a block occurring in X, approximating μ up to ϵ_n. We can assume that $|B| \gg N$. By definition of X, B must occur in $X_{h_y^{(m)}}$ for some $m > n$.

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It follows from the construction, that all symbolic sequences built after step n are infinite concatenations of the building N-blocks constructed in step n (perhaps with the first symbols changed later, which is negligible).

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Since the extremes do not depend on n, $\mu = \pi_\infty(y) \in \pi_\infty(K)$.

Almost done!

We have constructed a subshift X such that K is affinely homeomorphic (via π_∞) to $\mathcal{M}_T(X)$.

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THE END

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(in the order of appearance)

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