Ergodic Ramsey Theory

Vitaly Bergelson

The goal of the course is to introduce the participants to the main principles of Ergodic Ramsey Theory and to discuss some of the classical combinatorial and number theoretic results, such as van der Waerden, Hindman's and Szemeredi's theorems, from the point of view of topological dynamics, ergodic theory and topological algebra in Stone-Cech compactifications. Some recent developments and open problems will be surveyed.

The Multifarious Poincaré Recurrence Theorem

Vitaly Bergelson

We will discuss the origins of Poincaré recurrence theorem and its multiple connections and applications to physics, number theory and combinatorics. This lecture will serve also as the introduction to the mini-course Ergodic Ramsey Theory.

The Simplex of Invariant Measures of a Topological Dynamical System

Tomasz Downarowicz

We are interested in the set of invariant measures of a topological dynamical system, endowed with the weak* topology. This set is, first of all, nonempty, moreover it is compact and convex in a specific way that makes it a Choquet simplex. Its extreme points are precisely the ergodic measures. On the other hand, a topological analog of ergodicity is minimality. However, the analogy is far from perfect; it is known that minimal systems may support multiple ergodic measures. During the lecture we will show, among other things, that the variety of simplices of invariant measures appearing in minimal system is equally rich as the variety of simplices appearing in arbitrary topological dynamical systems.

Algebro-Geometric Methods in Classical Dynamics

Krzysztof Drachal

Fundamental concepts used in formal description of classical dynamics can be rewrtitten with help of algebra. They are e.g. phase space, manifold, vector field, differential equation, etc. There are reasonable arguments that this approach may lead to the precise mathematical description of some phenomens. Moreover algebraic approach gives the mentioned notions a meaning on structures more general than classical smooth manifold. But in fact real mathematical models has to cope with such structures (e.g. manifolds with boundary, signularities, etc.) - not with too idealized smooth manifolds. The above approach has evolved into the branch of mathematics called diffeotopy. Its objects of study are differential equations, theoretical physics, control theory, differential geometry and many others. As an illustrative example of this approach the Hamiltonian dynamics is given.

Entropy and Pythagoras' Theorem

Dariusz Dudzik

In thermodynamics entropy is often taken to be a measure of disorder, but in mathematics, especially in probability theory, it gives a measure of information.

The entropy functional introduced by Shannon and Wiener (independently) in 1948 is defined by

$$S(p) = -\sum_{i=1}^{n} p_i \ln p_i,$$

where p is a probability distribution with point masses $p_1, \ldots, p_n$. (It is referred to as the Shannon entropy functional).

In the talk we will present a few properties of the Shannon entropy functional, its extensions to general probability distributions and some real-life applications. We will also formulate and prove a Pythagorean-type theorem in a space of probability distributions.

Cylinder Flows with Transitivity, Discrete Orbits and Some Chaos

Eugeniusz Dymek

Cylinder flows are mappings of the form

$$X \times \mathbb{R} \ni (x,t) \mapsto (Tx, t + f(x)) \in X \times \mathbb{R}$$

where T is a homeomorphism of X and $f\colon X \to \mathbb{R}$. In the lecture we will usually consider transitive cylinders with some closed discrete orbits, and with T being a minimal rotation of a torus. We will show cylinders such that the set of discrete orbits, albeit in this setting always of measure zero, is of full Hausdorff dimension, or such that the function f is Hölder continuous. We will also recall a result of Frączek and Lemańczyk, which gives cylinders with both "large" set of discrete orbits and smooth f, but only for at least 3-dimensional tori (NB: this cannot be done for dimension one; the case of dimension two remains open). Finally, all these examples are sensitive to initial conditions and thus are chaotic in the sense similar to Devaney's: with 'periodic orbits' replaced by 'closed discrete orbits' in the definition of Devaney's chaos.

As an application, those constructions yield homeomorphisms of a plane and ODEs with similar properties.

Ergodic Theorems

Tanja Eisner

We discuss some generalisations and extensions of the classical ergodic theorems including weighted, subsequential and multiple ergodic theorems.

Characteristic Factors and Nilrotation

Mikołaj Frączyk

In order to investigate a measure preserving system X one often looks at its factors, which are simpler systems capturing some part of dynamics on X. Factors reflecting the behavior of Furstenberg type averages are called "n-characteristic factors", where n corresponds to the number of functions involved in the average. During the talk we will precisely state the definition of a n-characteristic factor and prove that the Kronecker factor is the 3-characteristic factor. Later we will introduce the class of dynamical systems called "nilrotations" which arise from rotations on quotients of nilpotent Lie groups. Finally we will be able to state the Host-Kra structure theorem which describes the characteristic factors as inverse limits of appropriate nilrotations.

Hausdorff and Spectral Dimension of Fractals

Uta Freiberg

Self similar fractals are often used in modeling porous media. Hence, defining a Laplacian and a Brownian motion on such sets describes transport through such materials. However, the assumption of strict self similarity could be too restricting. So, we present several models of random fractals which could be used instead. After recalling the classical approaches of random homogenous and recursive random fractals, we show how to interpolate between these two model classes with the help of so called V-variable fractals. This concept (developed by Barnsley, Hutchinson & Stenflo) allows the definition of new families of random fractals, hereby the parameter V describes the degree of "variability" of the realizations. We discuss how the degree of variability influences the geometric, analytic and stochastic properties of these sets. These results have been obtained with Ben Hambly (University of Oxford) and John Hutchinson (ANU).

If time allows, we will present the Einstein relation, expressing the interplay geometric, analytic and stochastic properties of a set. Applying this relation to the class of (non self similar) Hanoi attractors leads to a (theoretical) construction of "superconductors".

Ergodic Properties of Beta Expansions

Maciej Gawron

Let $\beta > 1$ be non-integer real number. We will consider digital expansions of nonnegative real number x in a form $x = a_0 + a_1\beta^{-1} + a_2\beta^{-2} + \ldots$ which is a generalization of common decimal expansion. We will describe properties of this expansions by studying a dynamical system $T_\beta : [0,1] \ni x \to \beta x \pmod 1 \in [0,1]$. We will prove that T_β has a unique invariant measure μ_β that is absolutely continous with respect to the Lebesgue measure on $[0,1]$. Furthermore, μ_β is ergodic and it is the unique measure of maximal entropy. We will discuss the problem of classification periodic β-expansions. We prove that $x \in [0,1)$ has periodic β expansion if, and only if $x \in \mathbb{Q}(\beta)$, when β is a Pisot number.

States vs. Observables - Operators in Ergodic Theory

Markus Haase

Ergodic theory was born when von Neumann and Birkhoff established their "Ergodic Theorems". The main protagonist in these results is not the space space dynamics but the so-called Koopman operator acting on observables. In my lectures I want to give an introduction to the operator theoretic side of ergodic theory. Furthermore, I shall give a fairly elementary proof of a famous result of Halmos-von Neumann-Rokhlin, namely that under natural assumptions the state space dynamics can be recovered from the Koopman operator.

Applications of Ultrafilters in Ergodic Theory

Jakub Konieczny

Ultrafilters are one of the most mysterious and surprising objects in mathematics. On the one hand, there is no explicit construction of an ultrafilter and even proof of their existence involves the axiom of choice. On the other hand, they turn out to have remarkable applications in a wide variety of branches of mathematics, including topology, analysis, and model theory. I will present some of the most interesting ways in which ultrafilters can be used in ergodic theory, for the purpose of deriving recurrence results.

On the Connection Between Hausdorff Dimension and Bowen's Definition of Topological Entropy

Martha Łącka

The usual topological entropy is defined only for maps of compact spaces. Nevertheless, there are many generalizations of this notion. One of them - introduced about 40 years ago by Bowen - concerns Lipschitz continuous maps of a compact space restricted to its any subset (non necessary compact) and is somewhat similar to Hausdorff dimension. In 2004 Misiurewicz used Bowen's definition of entropy in order to prove a theorem (orginally showed by Dai, Zhou and Geng) which says that if X is a metric compact space and $f : X \to X$ is a Lipschitz continuous map, then the Hausdorff dimension of X is bounded from below by the topological entropy of f divided by the logarithm of its Lipschitz constant. During the talk we will discuss this result.

Ergodic Theory and Continued Fractions

Marcin Lara

We will investigate the properties of so-called continued fraction map $T : [0,1] \setminus \mathbb{Q} \to [0,1] \setminus \mathbb{Q}, T(x) = \{\frac{1}{x}\}$. This map turns out to be ergodic with respect to the Gauss measure on $(0,1)$, which measure we will introduce during the lecture. We use this to deduce statements about the digits of the continued fraction expansion of a typical real number. For example: with what density does the digit j appear in the continued fraction of almost every real number. We will also see that there is a class of real numbers that behave very differently to typical ones.

Ergodic Theory and Diophantine Approximation

Przemysław Mazur

An easy application of the pigeonhole principle tells us that for any irrational number α there are infinitely many pairs of integers (p, q) with $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$. A more sophisticated argument can improve this bound to $\frac{1}{q^2\sqrt{5}}$. Unfortunately this is all we can get if we want to approximate well *all* numbers – the example $\alpha = \frac{1+\sqrt{5}}{2}$ shows that the latter bound is the best possible.

The question is: is this "badly approximable" number an exception or are there much more of them? In the talk I will try to answer it using ergodic theoretical methods. To be more precise, I will show that we can approximate *almost all* irrational numbers much better.

Population Dynamics and Ergodic Theorems

Piotr Szczepocki

The aim of the lecture is to present strong and weak theorems of population dynamics. These theorems have a different form than in classic ergodic theory. Term "ergodic" in demography refer to asymptotic property in which age distribution is independent of initial conditions. The lecture will demonstrate passages from demographic approach to dynamical systems and ergodic theory. The main tools to attain this purpose are Markov Chain theory and Garret Birkhoff theory of multiplicative processes.

Non-Archimedean Discrete Dynamical Systems

Anna Szumowicz

In my talk I will focus on Non-Archimedean discrete dynamical systems, especially on p-adic ergodic theory. One can study the ergodicity of a monomial dynamical systems in $\mathbb{Q}_p$. I will tell about measure-preserving and ergodic isometries on $\mathbb{Z}_p^n$. One of the results says that every ergodic isometry of $\mathbb{Z}_p$ is conjugated to the translation $g(x) = x + 1$ on $\mathbb{Z}_p$ .There are also some sufficient conditions (if $p = 2$, there are also necessary) expressed in terms of Mahler expansion of function f which allows us to say if function f is measure-preserving and 1-Lipschitz (or ergodic and 1-Lipschitz).

On Recurrence and Ergodicity for Geodesic Flows on Non-Compact Polygonal Surfaces

Anna Tyburska

The lecture will deal with recurrence and ergodicity for billiard flows on non-compact polygons and non-compact polygonal surfaces. We will present the ergodic decomposition of directional billiard dynamics in wind tree model for a dense, countable set of directions.

The lecture present the recent results of professors Conze and Gutkin on recurrence and ergodicity for billiard flows on non-compact polygons and non-compact polygonal surfaces.

Sated Systems and Multiple Ergodic Averages

Pavel Zorin-Kranich

Factors of measure-preserving systems have been extensively studied in connection with ergodic Ramsey theory. The usual approaches to convergence and recurrence for multiple ergodic averages rely on identifying some structure in these śmallerśystems. Extensions, on the other hand, have received less attention, probably because their structure was believed to be more complicated than that of the original systems. This changed with the work of Austin, who introduced sated extensions, whose existence is easy to establish and whose properties make them extremely convenient when dealing with multiple ergodic averages. I will present the proof of norm convergence of multiple ergodic averages within this framework and, if time permits, sketch the proof of Szemerédi's theorem which is only rivalled by the Polymath proof in its conciseness.